**REVIEW OF *THE MAKING OF MATHEMATICS – HEURISTIC PHILOSOPHY OF MATHEMATICS*, BY CARLO CELLUCCI**

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In 2022, Carlo Cellucci, Professor Emeritus at the University of Rome, published what can be understood to be one of the most significant books in the philosophy and pedagogy of mathematics in the past century. His erudition and explication are evident on every page, but what makes it most valuable for an understanding and presentation of mathematics is the clarity he brings to the argument that while mathematics has been considered the past century to be represented by theorem proving by the axiomatic method when considered more completely it is best understood as problem solving by the analytic method. Such an understanding could transform the way mathematics is presented in journals, textbooks, and classrooms.

Cellucci writes in the Preface that “Mainstream philosophy of mathematics . . .claims that the philosophy of mathematics cannot concern itself with the making of mathematics, in particular discovery, but only with finished mathematics. . . . This book offers an alternative approach, heuristic philosophy of mathematics, according to which the philosophy of mathematics can concern itself with the making of mathematics, in particular discovery.” In doing so, he reconnects the context of discovery with that of justification, so that the organic development of doing mathematics is made evident. In this way mathematical arguments are more complete in their communication to members of the mathematics community and provides students with a much more satisfactory understanding of what is involved in doing mathematics.

As he observes, “Mathematics is made starting from problems, formulating hypotheses for their solution by non-deductive inferences, and establishing their plausibility through a comparison with experience” (p. 436). To instantiate the distinction here the Law of Sines is considered. As presented as an exemplar of theorem proving, it presents a critical enduring problem for students – with their being presented with findings that omit consideration of the hypotheses that gave impetus for the initial exploration. As Dewey noted, “the mistake is, logically, due to the attempt to introductive deductive considerations without first making acquaintance with the particular facts that create a need for the generalizing rational devices” (1918, 98-99). In the instance considered here, a scalene triangle along with a perpendicular drawn to one side of the triangle is presented to the student. Then the proof is provided with finding the ratios of the sides associated with the sines of the angles. But what is missing is discussion regarding how would anyone have known to have drawn the perpendicular. More completely, what was the impetus for it? Here we have an instance of keeping the initial hypothesis in the shadow. We can imagine that in Plato’s Academy there was a search for any generalizations that could be made regarding triangles as Pythagoras’s Theorem was well known by that time. Clearly, there is no there there when looking at a general triangle with its three angles and three opposite sides; something needs to be done in the hope of finding any relationships. Drawing the perpendicular line segment creates that possibility. That is, the heuristic action of *tinkering* in an effort to make some connection is analogous to putting bait on a fish hook. And in the particular case for what can be said about the general triangle, with having drawn the perpendicular an interesting finding is indeed uncovered. But its presentation with the focus of theorem proving to establish *the Law of Sines* is bereft of any connection to the desire for solving the problem of gaining understanding with regard to the general triangle. So in its presented form, it makes the outcome something to be memorized not memorable.

As Cellucci critiques in general, “by omitting the real mathematical process, textbooks based on axiomatic demonstration misrepresent the nature of the subject. To represent the latter authentically, the real mathematical process should be included” (268). Toward reifying that experience, he reinvigorates and formalizes the consideration of heuristics which had been earlier associated with Polya’s practice as essential to problem solving. The challenge to the formalistic presentation model of mathematics in practice today is critically demonstrated by Cellucci. At its root, he points out that the notion that mathematics as theorem demonstration by the axiomatic method has been shown to be problematic by Godel’s theorems, and that the latter supports Cellucci’s argument that mathematics is essentially and practically problem solving by the analytic method.

It should be noted that such consideration is integral to all elements of mathematics, including definitions. He shows that the traditional stipulative view of mathematical definitions provides a poor understanding of what underlies the establishing of a definition. He offers the heuristic view – “namely, mathematical definitions are hypotheses which are made to solve mathematical problems by the analytic method. Like all hypotheses in the analytic method, such hypotheses must be plausible” (294). That is, although definitions are value-free with regard to their truth or falsity, as Russell shared, they are clearly value-based, being selectively chosen in an effort to further the body of mathematical knowledge. In “Manufacturing a Mathematical Group: A Study in Heuristics”, Ippoliti (2020) locates the dedicated investigatory effort involved in the foundational conceptual development of the definition of a mathematical group. He uncovers the heuristics of *look for similarities*, *change of representation*, *generalize from particulars*, and *reason by analogy* that Lagrange, Cauchy, Galois, and Cayley drew upon as essential actions in its development. As the reader can garner with regard to its formulation, “Focusing on the heuristics that gradually have led to its formation and refinement . . . displays paradigmatic features of the core of problem-solving” (ibid, 1). In stark contrast, the traditional textbook introduction of a mathematical group provides the definition accompanied by illustrative examples to make evident its mathematical value. Were all definitions presented in this manner, students would seemingly have little evidence toward understanding their constructed nature and essential role in furthering the body of mathematics knowledge.

If the mathematics community is open to seriously considering Cellucci’s extraordinary exposition, mathematics may well be rewritten in the decades to come. Where heuristic considerations inform the work and the reader of what is actually involved in doing mathematics.

References

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Ippoliti, Emiliano. 2018. Manufacturing a mathematical group: A study in heuristics. Springer

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